ON THE MULTIVALENCE AND ORDER OF STARLIKENESS OF A CLASS OF MEROMORPHIC FUNCTIONS

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ABSTRACT

Basically this paper deals with the determination of the radius of starlikeness and radius of univalence of the class of meromorphic functions of the form

$$g(z) = A/z - \phi(z)$$

for $A > 0$, and where $\phi(z)$ is an analytic function defined in the unit disk whose modulus does not exceed unity. We estimate the radius of p-valence of functions having the form

$$h(z) = \phi(z) + a/z^p$$

for $a > 1$, $p \geqq 1$, and also estimate the radius of starlikeness of certain Blaschke products which is also given as a function of the minimum modulus function. We discuss the question of sharpness of the results and mention some open problems.

1. Introduction

In this paper we consider meromorphic functions in the form

$$f(\zeta) = \sum_{k=1}^{p} A_k/(\zeta - a_k), \ A_k > 0, \ \sum_{k=1}^{p} A_k = 1, \ |a_k| < 1, \tag{1}$$

as well as the more general class of functions

$$g(z) = A/z - \phi(z), \tag{2}$$

$A > 0$ and $\phi(z)$ an analytic function defined in the unit disc, such that $|\phi(z)| \leqq 1$ for $|z| < 1$. These classes are closely related to problems in the theory of

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polynomials and were studied by the authors of [2], [3], [4], [5], [10], [11] and [13].

The relation between functions (1) and (2) was pointed out by the first author in [9] by means of a result obtained in [10].

Throughout the paper we will denote by B_1 the class of functions $\Phi(\zeta)$ analytic and $|\Phi(\zeta)| \leqq 1$ for $|\zeta| > 1$; and by B_2, the corresponding class of analytic functions $\phi(z)$, $|\phi(z)| \leqq 1$ in $|z| < 1$.

According to [10, Th. C], we can write (1) in the form

$$f(\zeta) = \frac{1}{\zeta - \Phi(\zeta)}, \quad \Phi(\zeta) \in B_1, \quad |\zeta| > 1. \tag{1'}$$

Representation (3) below which is equivalent to (1′) is obtained by substituting $\zeta = 1/z$;

$$f(z) = \frac{z}{1 - z\phi(z)}, \quad \phi(z) \in B_2, \quad |z| < 1. \tag{3}$$

For the special case $A = 1$ in (2), we then have the relation $g(z) = 1/f(z)$. In Section 2 we determine the radius of starlikeness and univalence for our classes of functions.

In Section 3 a previous result by the second author [12] is applied to obtain an estimate of the order of starlikeness of analytic functions defined in the unit disc whose modulus does not exceed one. This estimate depends solely on the behavior of the minimum modulus function $m(\rho)$. Section 4 deals with the determination of the order of starlikeness for the various classes and these results are sharpened for functions with additional symmetry conditions. Finally, an estimate of the radius of p-valence of functions of the form

$$h(z) = \phi(z) + A/z^p, \tag{4}$$

where A is positive and $\phi(z \in) B_1$, is derived. All estimates are shown to be sharp for small values of the parameter A. We also make some remarks on the question of convexity of the above-mentioned classes of functions.

2. Radii of starlikeness

In this section we will determine the regions of starlikeness for our functions. The first theorem deals with the radius of starlikeness for function (1) and function (2) with $A = 1$. The proof is based on a result of MacGregor [7] for analytic functions. In the second theorem the radius of starlikeness for functions (2) with arbitrary constant A is derived.

THEOREM 2.1. *Function* (3) *is starlike for* $|z| < 2^{-\frac{1}{2}}$ *and function* (1) *is starlike for* $|\zeta| > 2^{\frac{1}{2}}$.

PROOF. It follows from representation (3) that the function $f(z)/z$ is subordinate to $1/(1-z)$. Therefore $\operatorname{Re} f(z)/z > \frac{1}{2}$; $f(z)/z = 1$ for $z = 0$ and by a theorem of MacGregor [7, p. 74] $f(z)$ is univalent and starlike for $|z| < 2^{-\frac{1}{2}}$. The result is sharp. The function $g(z) = 1/f(z) = 1/z - (1 - 2^{\frac{1}{2}}z)/(2^{\frac{1}{2}} - z)$ has $g'(2^{-\frac{1}{2}}) = 0$, and $F(\zeta) = 1/f(\zeta) = \zeta - (\zeta - 2^{\frac{1}{2}})/(\zeta 2^{\frac{1}{2}} - 1)$ has $F'(2^{\frac{1}{2}}) = 0$.

THEOREM 2.2. *Function* (2) *is univalent and starlike in the disc*

$$|z| < A^{\frac{1}{2}}/(1+A)^{\frac{1}{2}} \text{ for } A \geqq (5^{\frac{1}{2}} - 1)/2 \tag{5}$$

$$|z| < A \text{ for } A \leqq (5^{\frac{1}{2}} - 1)/2 \tag{6}$$

and the result are sharp.

PROOF. By differentiation of (2) we obtain

$$\frac{-zg'(z)}{g(z)} = \left(1 + \frac{z^2\phi'(z)}{A}\right) \Big/ \left(1 - \frac{\phi(z)}{A}\right).$$

To prove the starshapedness of $g(z)$ we need to show that

$$\left|\arg \left(1 + \frac{z^2\phi'(z)}{A}\right) \Big/ \left(1 - \frac{\phi(z)}{A}\right)\right| < \frac{\pi}{2},$$

that is,

$$|\arg(1 + z^2\phi'(z)/A| + |\arg(1 - \phi(z)/A| < \pi/2.$$

Let $|z| < \min(A, (A/(1+A))^{\frac{1}{2}})$. This is equivalent to conditions (5) and (6). It follows that $|z\phi(z)/A| < 1$, and using a well-known inequality for $|\phi'(z)|$ we obtain

$$\frac{|z^2\phi'(z)|}{A} \leqq \frac{|z^2(1 - |\phi(z)|^2|}{A(1 - |z|^2)} \leqq 1 - |\phi(z)|^2 \text{ for } |z| \leqq (A/(1+A))^{\frac{1}{2}}.$$

These conditions imply that

$$|\arg - zg'(z)/g(z)| \leqq \sin^{-1}(1 - |\phi|^2) + \sin^{-1}|\phi|.$$

We have $\sin^{-1}|\phi| = \cos^{-1}[1 - |\phi|^2]^{\frac{1}{2}} \leqq \cos^{-1}(1 - |\phi|^2) = \frac{1}{2}\pi - \sin^{-1}(1 - |\phi|^2)$. It follows that $|\arg[-z\, g'(z)/g(z)]| \leqq \frac{1}{2}\pi$.

To see that the results are sharp consider the following examples. For

$A \leqq (5^{\frac{1}{2}} - 1)/2$, let $g(z) = A/z + 1$. The point $z = -A$, is mapped into the origin. The image of a circle $|z| > A$ will not include the origin and so is not starshaped with respect to zero. For $A \geqq (5^{\frac{1}{2}} - 1)/2$, let

$$g(z) = A/z - (b - z)/(1 - bz), \; b = [A/(1 + A)]^{\frac{1}{2}}.$$

Computation will verify that $g'(b) = 0$.

REMARK. The condition for univalence $|z| < [A/(1 + A)]^{\frac{1}{2}}$ was established in [9]. The above example shows that it is sharp.

The conditions for starlikeness of function (1) and (2) can be sharpened if additional symmetry properties are imposed on the functions. This will follow as a consequence of the results of Section 4.

3. Blaschke products

We begin this section with an estimate of the order of starlikeness of analytic functions which depends only on the behavior of the minimum modulus function. This estimate uses the inequality established in [11] and restated below for convenience.

THEOREM 3.1. *Let $f(z)$ be regular in the unit disk and of modulus not exceeding one there. Let $g(z) = \sum_{k=p}^{\infty} a_k z^k$, $p \geqq 1$. Then $|g'(z)| \leqq p|z|^{p-1}$ for $|z| \leqq [(1 + p^2)^{\frac{1}{2}} - 1]/p$ and*

$$|g'(z)| \leqq |z|^{p-2}[4|z|^2 + p^2(1-|z|^2)^2]/4(1-|z|^2) \text{ for } |z| > ((1+p^2)^{\frac{1}{2}}-1)/p.$$

We are now ready to prove Theorem 3.2.

THEOREM 3.2. *Let $f(z) = \sum_{k=0}^{\infty} c_k z^k$ be analytic in the unit disk and of modulus not exceeding one there. Then*

$$\operatorname{Re} e^{i\theta} f'(e^{i\theta})/f(e^{i\theta}) \leqq \limsup_{\rho \to 1} [1 - m(\rho)/m(\rho)(1 - \rho)] \text{ where } m(\rho) = \min_{|z| = \rho} |f(z)|,$$

at any point $e^{i\theta}$ where $f(z)$ and $f'(z)$ are continuous.

PROOF. We apply the previous theorem mentioned above to the functions $g_p(z) = z^p f(z)$ for $p = 1, 2, 3, \cdots$. It follows that $|f(z)| \, |p + zf'(z)/f(z)| \leqq p$ or $|z| = r_p$ where $r_p = [(1 + p^2)^{\frac{1}{2}} - 1]/p$. Hence $\operatorname{Re} zf'(z)/f(z) \leqq p\,(1/m(r_p) - 1)$ on $|z| = r_p$. Since $p(1/m(r_p) - 1) = (1/(1 - r_p))(1/m(r_p) - 1)\, p(1 - r_p)$, $\lim_{p\to\infty} p(1 - r_p) = 1$, $\lim_{p\to\infty} r_p = 1$ we have $\limsup_{p\to\infty} p(1/m(r_p) - 1) = \limsup_{p\to\infty} [(1/(1-\rho)(1/m(\rho) - 1)]$. Now let $z_p = r_p e^{i\theta} p$ be a sequence of points such that $\lim_{p\to\infty} z_p = e^{i\theta}$ and $f(z_p) \neq 0$, then since $\operatorname{Re} z_p f'(z_p)/f(z_p) \leqq p(1/m(r_p) - 1)$, the result follows letting p tend to infinity.

COROLLARY 3.3. *Let $f(z)$ be as in Theorem 3.2 and such that $f(z)$ is continuous in $|z| \leqq 1$ and $|f(e^{i\theta})| = 1$, $0 \leqq \theta \leqq 2\pi$. Then*

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \leqq \limsup_{\rho\to 1} \frac{1-m(\rho)}{1-\rho} \text{ for } |z| = 1.$$

Obviously the above result is applicable for Blaschke products. In this connection it is worthwhile to mention a result due to Stephen Fisher [6], namely, that every function analytic in the unit disk, continuous in its closure and of modulus not exceeding one is uniformly approximable by convex combinations of finite Blaschke products. Theorem 3.2, therefore, provides additional information on the behavior of the approximating sequence and the approximable function and its derivative.

4. The order of starlikeness of meromorphic functions

In this section we will establish the order of starlikeness, first for functions (1) and (3), and secondly for functions (2) with arbitrary A.

THEOREM 4.1. *Function* (1) *is starlike of order* α, $0 \leqq \alpha \leqq 1$ *where*

$$\alpha = 2R\frac{\{[2(R^2-1)]^{\frac{1}{2}} - R\}}{R^2-1} \text{ for } 2^{\frac{1}{2}} \leqq R = |\zeta| < 3 \tag{7}$$

$$\alpha = R/(R+1) \text{ for } R \geqq 3. \tag{8}$$

Function (3) *is starlike of order*

$$\alpha = 2\frac{\{[2(1-r^2)]^{\frac{1}{2}} - 1\}}{1-r^2} \text{ for } \tfrac{1}{3} \leqq |z| = r \leqq 2^{-\frac{1}{2}} \tag{9}$$

$$\alpha = 1/(1+r) \text{ for } r \leqq \tfrac{1}{3}. \tag{10}$$

PROOF. Let $F(\zeta) = 1/f(\zeta)$ and define the functions S by

$$S(\zeta) = -\operatorname{Re}\frac{\zeta f'(\zeta)}{f(\zeta)} = \operatorname{Re}\zeta\frac{F'(\zeta)}{f(\zeta)} \tag{11}$$

$$S(z) = \operatorname{Re}\frac{zf'(z)}{f(z)} = -\operatorname{Re}\frac{zg'(z)}{g(z)}. \tag{12}$$

By differentiation of (1′) and (3) we obtain the following equivalent representations for the function S.

$$S(\zeta) = \operatorname{Re}\frac{\zeta(1-\Phi'(\zeta))}{\zeta - \Phi(\zeta)}, \text{ for } |\zeta| > 1,\ \Phi(\zeta) \in B_1 \tag{13}$$

(14) $$S(z) = \operatorname{Re} \frac{1 + z^2\phi'(z)}{1 - z\,\phi(z)} \text{ for } |z| < 1,\ \phi(z) \in B_2.$$

We will carry out the proof of our theorem for representation (13). The result for function (3) will follow by substituting $z = 1/\zeta$.

For our proof we must determine

$$\min_{\substack{|\zeta| = R \\ \Phi(\zeta) \in B_1}} S(\zeta)$$

Let $|\zeta - \Phi(\zeta)| = s$; let $\arg\,(\zeta - \Phi(\zeta)/\zeta) = \delta$. For $|\zeta| > 1$, $\Phi(\zeta) \in B_1$ and $|\delta| \leqq \sin^{-1} 1/|\zeta|$. We have $S(\zeta) = \operatorname{Re}\zeta/(\zeta - \Phi(\zeta)) - \operatorname{Re}\zeta\Phi'(\zeta)/(\zeta - \Phi(\zeta)) \geqq R/s \cos\,\delta - (R/s)\,|\Phi'(\zeta)|$, where $|\zeta| = R$. For $\Phi(\zeta) \in B_1$ we have

$$|\Phi'(\zeta)| \leqq (1 - |\Phi(\zeta)^2)/(R^2 - 1)$$

and we obtain

(15) $$(R^2 - 1)S(\zeta)/R \geqq [\cos\delta(R^2 - 1) - (1 - \Phi(\zeta)|^2)]/s.$$

The proof proceeds by expressing $|\Phi(\zeta)|^2$ in terms of s and δ, that is, $|\Phi(\zeta)|^2 = s^2 + R^2 - 2Rs\cos\delta$.

Substituting in (15) and simplifying it follows that

(16) $$S(\zeta)/R \geqq \frac{\cos\delta + 1}{s} + \frac{s}{R^2 - 1} - \frac{2R\cos\delta}{R^2 - 1}.$$

The right-hand side of (16) is decreasing with respect to $\cos\,\delta$ if $1/s - 2R/(R^2 - 1) < 0$. This condition is satisfied for all possible values of $s, s > R - 1$, and $R > 1$. As a consequence the smallest lower bound for $S(\zeta)$ is obtained for $\cos\,\delta = 1$.

(17) $$S(\zeta)/R \geqq 2/s + s/(R^2 - 1) - 2R/(R^2 - 1).$$

The right-hand side of 17 assumes its minimum value for $s = s_{\min} = [2(R^2 - 1])^{\frac{1}{2}}$. The value (7) for the order of starlikeness α is obtained when $s_{\min}$ is a possible value of s, that is, if $s_{\min} \leqq R + 1$. Value (8) follows for $s = R + 1$.

The values (8) and (9) are obtained by substituting $r = 1/R$.

REMARK. The order of starlikeness was defined for positive α but the negative value of α obtained for $R < 2^{\frac{1}{2}}$ and $r > 2^{-\frac{1}{2}}$ are equally valid.

The results are sharp. Direct computation will verify that the extremal functions are $1/f(\zeta) = F(\zeta) = \zeta + 1$ and $F(\zeta) = \zeta - (\zeta b - 1)/(\zeta - b)$ or $g(z) = 1/z + (1 - z/b)/(z - 1/b)$ where $R - (Rb - 1)/(R - b) = [2(R^2 - 1)]^{\frac{1}{2}}$, that is,

$$b = \frac{R^2 + 1 - R[2(R^2 - 1)]^{\frac{1}{2}}}{2R - [(R^2 - 1)]^{\frac{1}{2}}}.$$

The next result deals with function (2). The methods of the above proof can be generalized to determine the order of starlikeness for function (2) with arbitray $A \geqq 1$.

THEOREM 4.2. *Function* (2), $A \geqq 1$, *is starlike of order* α, $0 \leqq \alpha \leqq 1$ *where*

$$\alpha = \frac{2\{[A - r^2)(1 + A)]^{\frac{1}{2}} - A\}}{1 - r^2} \tag{18}$$

for

$$\left[\frac{A + 1}{A}\right]^{-\frac{1}{2}} \geqq r \geqq \frac{[2(1 + A)A]^{\frac{1}{2}} - A}{2 + A} = r_A$$

$$\alpha = \frac{A}{A + r} \text{ for } r \leqq r_A. \tag{19}$$

PROOF. Let $S(z) = -\operatorname{Re} z g'(z)/g(z)$.

By differentiating (2),

$$S(z) = \operatorname{Re} \frac{A + z^2\phi'(z)}{A - z\phi(z)} \text{ for } |z| < 1,\ \phi(z) \in B_2 \text{ or} \tag{20}$$

$$S(\zeta) = \operatorname{Re} \frac{\zeta[A - \Phi'(\zeta)]}{A\zeta - \Phi(\zeta)} \text{ for } |\zeta| > 1,\ \Phi(\zeta) \in B_1. \tag{21}$$

Following the proof of the previous theorem we are going to determine

$$\min_{\substack{|\zeta| = R \\ \Phi(\zeta) \in B_1}} S(\zeta).$$

Let $|A\zeta - \Phi| = s$, $\arg(A - \Phi/\zeta) = \delta$.

Assume $A|\zeta| > 1$, $|\delta| \leqq \sin^{-1} 1/(A|\zeta|)$

$$S(\zeta) = \operatorname{Re} \frac{A\zeta}{A\zeta - \Phi(\zeta)} - \operatorname{Re} \frac{\zeta\Phi'(\zeta)}{A\zeta - \Phi(\zeta)} \geqq \frac{AR\cos\delta}{s} - \frac{R|\Phi'(\zeta)|}{s}$$

$$(R^2 - 1)\, S(\zeta)/R \geqq \frac{A(R^2 - 1)\cos\delta - (1 - |\Phi(\zeta)|^2)}{s}$$

$$|\Phi(\zeta)|^2 = A^2R^2 + s^2 - 2ARs\cos\delta,$$

$$\frac{R^2 - 1}{R} S(\zeta) \geqq \frac{A(R^2 - 1)\cos\delta - 1 + A^2R^2}{s} + s - 2AR\cos\delta.$$

The right-hand side decreases with $\cos\delta$ since $A(R^2 - 1)/s - 2AR < 0$. In order to find the minimum of $S(\zeta)$ we again replace $\cos\delta$ by 1

$$\frac{(R^2-1)\,S(\zeta)}{R} \geqq \frac{(AR^2-1)\,(1+A)}{s} + s - 2AR.$$

The minimum value of the function on the right-hand side is assumed for

$$s_{\min} = [(AR^2-1)\,(1+A)]^{\frac{1}{2}}.$$

This is a possible value of s if $R < 1 + [2(1+A)/A]^{\frac{1}{2}}$. The value $s = AR + 1$ is substituted in the other case. The orders of starlikeness (18) and (19) are obtained by substituting $r = 1/R$.

The order of starlikeness for functions $g(z)$ with $A < 1$ is obtained by the same method, but the auxiliary condition $|z| < A$ must be added to the other conditions.

The results are sharp. Computation will verify the order of starlikeness for $|z| = 1/R$ of the function $g(z)\,A/z + (1-z/b)/(z-1/b)$ where b is given by the solution of

$$AR - (Rb-1)/(R-b) = [(AR^2-1)\,(1+A)]^{\frac{1}{2}}, \text{ and } g(z) = A/z + 1.$$

5. The order of starlikeness for functions with symmetry conditions

The order of starlikeness for function (1) calculated in Section 4 can be improved if conditions of symmetry are imposed on the poles a_k. As a consequence, the estimates of the radii of starlikeness are also sharpened.

THEOREM 5.1. *In addition to the hypotheses of Theorem* 4.1, *assume* $\Sigma_1^n a_k^l = 0$ *for* $l = 1, 2, \cdots, n$. *Then the order of starlikeness* α *of function* (1) *is*

$$(22)\qquad \alpha = \frac{2R[R^{2n+2} - 1 + (1+n)\,(R^2-1)R^n]^{\frac{1}{2}} - 2R^{n+2} - n(R^2-1)}{R^2-1},$$

$$\textit{for } |\zeta| = R < R_0$$

$$(23)\qquad \alpha = \frac{R^{n+1} - n}{R^{n+1} + 1} \textit{ for } R > R_0, \textit{ where } R_0 \textit{ is the solution of}$$

$$2[R^{n+1} + 1] = (n+1)R^n(R^2-1).$$

PROOF. By an improved version of the coincidence lemma [10] the symmetry condition on the a_k leads to the condition $|\Phi(\zeta)| < |\zeta|^{-n}$. We can write (13) in the form

$$S(\zeta) = \mathrm{Re}\left[\frac{\zeta}{\zeta - \Phi}\right] - \mathrm{Re}\left[\frac{\zeta}{\zeta - \Phi(\zeta)}\left(\Phi' + \frac{n\Phi}{\zeta}\right)\right] + \mathrm{Re}\left[\frac{n\Phi}{\zeta - \Phi}\right].$$

With the notation introduced in the proof of Theorem 4.1,

$$S(\zeta) \geqq \frac{R}{s}\cos\delta - \frac{R}{s}\left|\Phi' + \frac{n\Phi}{\zeta}\right| + n\left(\frac{R}{s}\cos\delta - 1\right)$$

$$= \frac{(n+1)R\cos\delta}{s} - n - \frac{R}{s}\left|\Phi' + \frac{n\Phi}{\zeta}\right| .$$

We have $|\zeta^n\Phi(\zeta)| \leqq 1$. It follows that

$$|\Phi' + n\Phi(\zeta)/\zeta| \leqq (1 - |\Phi(\zeta)|^2 R^{2n})/(R^n(R^2 - 1)).$$

Following the same methods of the proof of Theorem 4.1 we obtain the inequality

$$R^{n-1}(R^2 - 1)\,S(\zeta)$$

$$\geqq ((n+1)R^n(R^2 - 1)\cos\delta - 1 + R^{2n+2})/s + sR^{2n} - 2R^{2n+1}\cos\delta - nR^{n-1}(R^2 - 1).$$

The condition for the coefficient of $\cos\delta$ to be negative is

$$[(n+1)R^n(R^2 - 1)]/s < 2R^{2n+1}.$$

By the coincidence lemma, $s > R - 1/R^n$, and we obtain the condition $2R(R^{n+1} - 1) > (n+1)(R^2 - 1)$, or

$$2R[R^n + R^{n-1} + \cdots + 1] > (n+1)(R+1).$$

We have $2R > R + 1$, each term enclosed in the square brackets is greater than or equal to one and the condition is satisfied. We obtain

$$R^{n-1}(R^2 - 1)\,S(\zeta)$$

$$\geqq ((n+1)R^n(R^2 - 1) + R^{2n+2} - 1)/s + sR^{2n} - 2R^{2n+1} - nR^{n-1}(R^2 - 1).$$

From here the result follows by established methods.

COROLLARY 5.2. *Under the hypothesis of the theorem, the radius of starlikeness r_s is given by the smallest positive solution of*

$$4(R^2 - 1)\,R^{n+2} = n^2(R^2 - 1)^2 + 4R^2.$$

PROOF. This follows, after simplification, by setting in Theorem 5.1 the condition $\alpha \geqq 0$.

COROLLARY 5.3. *Consider function* (4), $h(z) = \phi(z) + 1/z$ *where* $|\phi(z)| \leqq |z|^n$. *The order of starlikeness for this function is given by*

$\alpha = (2[1 - r^{2n+2} + (1+n)(1 - r^2)r^n]^{\frac{1}{2}} - 2 - n(1 - r^2)r^n)/(1 - r^2)r^n$ *for* $r > r_0$,

$\alpha = (1 - nr^{n+1})/(1 + r^{n+1})$ *for* $r \leqq r_0$ *where* r_0 *is the solution of* $2r[1 + r^{n+1}] = (n+1)(1 - r^2)$. *The radius of starlikeness* r_s *of* (4) *is given by the smallest possible solution of* $4r^{n+2} = 4(1 - r^2) - n^2(1 - r^2)^2 r^n$.

6. Radius of multivalence of functions of type (3)

It turns out that similar methods to those employed in Section 2 can be extended to the question of the multivalence of meromorphic functions representable in form (3) which were introduced in Section 1. In this connection we quote the following theorem due to S. Ozaki [8].

THEOREM 6.1 (Ozaki). *If $f(z)$ is a meromorphic function which does not vanish in the disk $|z| \leqq r$, has exactly p poles inside this disk and none on the circumference, and if there exists a real number α such that* $\mathrm{Re}\{e^{i\alpha}(f'(z)/f(z))\} < 0$ *for $|z| = r$, then $f(z)$ is p-valent in the disk $|z| \leqq r$.*

See also M. Biernacki [1, p. 47]. We shall prove the following.

THEOREM 6.2. *Let $f(z) = \phi(z) + A/z^p$, $A > 0$, $p \geqq 1$, where $\phi(z) \in B_2$. Then $f(z)$ is p-valent in the disk $|z| \leqq \min(A^{1/p}, r_p)$ where r_p is the positive root of the equation $x^{p+1} + Apx^2 - Ap = 0$.*

PROOF.
$$\frac{-zf'(z)}{f(z)} = p - \frac{\phi'(z)z^{p+1}}{A} \Big/ \left(1 + \frac{\phi(z)z^p}{A}\right).$$

On
$$|z| = r, \quad \left|\frac{\phi'(z)z^{p+1}}{A}\right| \leqq \frac{1 - |\phi(z)|^2}{1 - r^2}\,\frac{r^{p+1}}{A}$$
and
$$\left|\frac{\phi(z)z^p}{A}\right| \leqq \frac{|\phi(z)|r^p}{A}.$$

For $|\alpha| < 1$ and $|\beta| < 1$ the condition $\mathrm{Re}[(1+\alpha)/(1+\beta)] > 0$ or
$$|\arg[(1+\alpha)/(1+\beta)]| < \tfrac{1}{2}\pi$$
will be satisfied whenever
$$|\arg(1+\alpha)| + |\arg(1+\beta)| < \tfrac{1}{2}\pi.$$
But $|\arg(1+\alpha)| \leqq \arcsin|\alpha|$, $|\arg(1+\beta)| \leqq \arcsin|\beta|$. Hence if
$$|\alpha|^2 + |\beta|^2 < 1,$$
$$|\arg(1+\alpha) + |\arg(1+\beta)| \leqq \arcsin|\alpha| + \arcsin|\beta|$$
$$< \arcsin(1 - |\beta|^2)^{\frac{1}{2}} + \arccos(1 - |\beta|^2)^{\frac{1}{2}} = \tfrac{1}{2}\pi.$$
This means it is sufficient to satisfy the inequality
$$\frac{1 - |\phi(z)|^2}{(1 - r^2)^2}\,\frac{r^{2(p+1)}}{A^2p^2} + \frac{|\phi(z)|^2 r^{2p}}{A^2} \leqq 1$$

except for the case when $\phi(z)$ is a linear transformation where a strict inequality is required.

One verifies readily that the last inequality will be satisfied whenever

$$\frac{r^{(p+1)}}{Ap(1-r^2)} \leqq 1 \text{ and } \frac{r^p}{A} \leqq 1.$$

Hence the result since the function $f(z)$ cannot vanish inside the disk of radius $\min(r_p, A^{1/p})$.

For $p = 1$ we obtain the result of Theorem 2.2. For $p > 1$ the functions $f(z) = 1 + A/z^p$ shows that the estimate is sharp whenever $A^{1/p} < r_p$. Moreover the function

$$f(z) = e^{i\beta}\left(\frac{z-\bar{\alpha}}{1-\alpha z}\right) + \frac{A}{z^p}, \ \alpha = r_p e^{i\beta/(p+1)}$$

has a critical point at α, which shows that there exist functions of type (3) which have a critical point on the circumference $|z| = r_p$ for arbitrary p. On the other hand it is clear from the estimates made in the proof of the theorem that unless $\phi(z) \equiv 1$, only when $\phi(z)$ is a linear function can we have $\mathrm{Re}(zf'(z)/f(z)) = 0$ on the circle $|z| = \min(A^{1/p}, r_p)$. Therefore $\phi(z) \equiv 1$ and $\phi(z) = e^{i\beta}(z-\bar{\alpha})/(1-\alpha z)$ are the only possible $\phi(z)$ for the extremal function.

For the case $p = 1$ these remarks yield another proof for a theorem due to P.G. Todorov [13], namely: the function

$$h(\zeta) = \sum_{k=1}^{p} A_k/(\zeta - a_k), \ p \geqq 2, \ A_k > 0, \ \sum_{k=1}^{p} A_k = 1, \ |a_k| \leqq 1$$

is univalent in the region $|\zeta| \geqq 2^{\frac{1}{2}}$ except when $p = 2$, in which case there exists a function of the above class for which the derivative $h(\zeta)$ vanishes on the circle $|\zeta| = 2^{\frac{1}{2}}$.

To see this one has only to write (3) in the form

$$\frac{1}{f(z)} = \frac{1}{z} - \phi(z), \ f(z) = h(\zeta), \ \zeta = \frac{1}{z}.$$

It follows by the previous argument that $h(\zeta)$ is univalent in $|\zeta| \geqq 2^{\frac{1}{2}}$ unless the function $\phi(z)$ is a linear function or a constant. This implies $p = 1$ or $p = 2$ and by hypothesis $p = 2$.

7. Some final remarks

One can extend the investigation of the classes of function studied in the previous sections. For instance one can ask whether there exists a radius of convexity (or close-to-convexity) for the classes studied previously. We shall limit ourselves to the following.

THEOREM 7.1. *Consider the function* $f(z) = \phi(z) + A/z$, $A > 0$, $\phi(z) \in B_2$. *There exists a number* r_A *which depends only on* A *such that the function* $f(z)$ *is convex on the circle* $|z| = r$ *for all* $0 < r < r_A$.

PROOF. One looks for r such that $-\operatorname{Re}(1 + zf''(z)/f'(z)) > 0$ on $|z| = r$. This amounts to

$$\text{(24)} \qquad \operatorname{Re}\left[\left(1 + \frac{z^2}{A}(z\phi'(z))'\right)\right] \Big/ \left(1 - \frac{z^2}{A}\phi'(z)\right) > 0.$$

Define the function $h(z) = z\phi'(z)$. Then $|h(z)| \leqq 1$ for $|z| \leqq \rho$, where $\rho = (5^{\frac{1}{2}} - 1)/2$.

The function $H(z) = h(\rho z) \in B_2$; by estimating the modulus of its derivative one obtains $|h'(z)| \leqq \rho(1 - |h(z)|^2)/(\rho^2 - |z|^2)$ for $|z| < \rho$. As in Section 6, condition (24) will be satisfied if $|z|^4 |z\phi'(z)|^2 + |z|^4 |\phi'(z)|^2 < A^2$ for $|z| = r$. For $|z| < \rho$ it is, thus, enough to satisfy the inequality

$$r^4 \frac{\rho^2}{(\rho^2 - r^2)^2}(1 - |h(z)|^2) + r^4 \frac{(1 - |\phi(z)|^2)^2}{(1 - r^2)^2} < A^2.$$

One considers the simpler inequality

$$r^2 \frac{\rho^2}{(\rho^2 - r^2)^2} + r^4 \frac{1}{(1 - r^2)^2} < A^2 \text{ with } 0 < r < \rho.$$

Denoting $R = r^2, \alpha = \rho^2 = (3 - 5^{\frac{1}{2}})/2$ one obtains

$$R^2 \frac{\alpha}{(\alpha - R)^2} + R^2 \frac{1}{(1 - R)^2} < A^2.$$

Since $\alpha/(\alpha - R)^2 > 1/(1 - R)^2$ it is sufficient to satisfy the inequality

$$\frac{2\alpha R^2}{(\alpha - R)^2} < A^2 \text{ or } r < \rho\left(\frac{A}{A + \rho(2)^{\frac{1}{2}}}\right)^{\frac{1}{2}}.$$

We were not able to find the exact value of r_A.

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